

# Effective mass schrödinger equation for exactly solvable class of one-dimensional potentials

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We deal with the exact solutions of Schrödinger equation characterized by position-dependent effective mass via point canonical transformations. The Morse, Pöschl-Teller, and Hulthén type potentials are considered, respectively. With the choice of position-dependent mass forms, exactly solvable target potentials are constructed. Their energy of the bound states and corresponding wavefunctions are determined exactly.

**KEY WORDS:** position-dependent mass, point canonical transformation, effective mass schrödinger equation, morse potential, pöschl-teller potential, hulthén potential

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## 1. Introduction

There has recently been a continual interest to the solutions of Schrödinger equations describing systems with spatially dependent effective masses [1–11]. Systems with certain types of effective masses are found to be useful in the determination of physical properties of semiconductor heterostructures [12], quantum liquids and dots [13, 14], helium and metal type clusters [15, 16], and nuclei [17] as well. In addition to these, the quantum mechanics for systems of particle with some types of masses has not only been considered for the instantaneous Galilean invariance [18], but also studied for the path integral approach [19] and  $su(1, 1)$  Lie algebraic technique, etc. [20, 21]. No matter how difficult exact solutions of are to obtain, the effective mass Schrödinger equation for certain type potentials can be solved [22, 23]. Within the framework of SUSYQM, problems including the shape invariance requirement are solved exactly [24–27].

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In the mapping of the nonconstant mass Schrödinger equation, point canonical transformations (PCTs) are employed [28–31]. When dealing with this process, we need to transform it to a constant mass one so that the latter equation can be solved easily. Therefore, it will yield both the energy spectra and corresponding wavefunctions of the target problem with regard to those of reference problem. Some reference (original) potentials satisfying the concept of exactly solvability such as oscillator, Coulomb, Morse [1], hard-core potential [32], trigonometric type [33] and conditionally exactly solvable potentials [2, 31] as well as the Scarf and Rosen-Morse type [34] ones including the PT-symmetry are considered for the construction of exact solution via PCT. In the present work, PCT approach is applied to the exact solutions of the nonconstant mass Schrödinger equation for Morse, Pöschl-Teller, and Hulthén potentials, respectively.

The plan of the article is in the following. In section 2, we show how to map the effective mass Schrödinger equation by using PCT method. In section 3, we apply the procedure to construct the target problem, including the energy spectrum and corresponding wavefunctions, for potentials mentioned above. Finally, we draw some remarkable conclusions in Section 4.

## 2. Mapping of the effective mass Schrödinger equation

The one-dimensional time independent Schrödinger equation within the case of spatially dependent mass is written

$$-\frac{1}{2} \left[ \nabla_x \frac{1}{M(x)} \nabla_x \right] \Psi(x) - [E - V(x)]\Psi(x) = 0, \tag{1}$$

where  $M(x) = m_0 m(x)$ . More explicit form of this equation is

$$\Psi''(x) - \left( \frac{m'}{m} \right) \Psi'(x) + 2m[E - V(x)]\Psi(x) = 0, \tag{2}$$

where we set  $\hbar = 1$  and  $m_0$  as constant, prime and double prime factors indicate the first and second order derivatives with respect to  $x$ . However, the one-dimensional Schrödinger equation with a constant mass can be written as

$$\Phi''(y) + 2[\varepsilon - V(y)]\Phi(y) = 0. \tag{3}$$

By introducing a transformation  $y \rightarrow x$  through a mapping function  $y = f(x)$  and rewriting the wavefunction

$$\Phi(y) = g(x)\Psi(x). \tag{4}$$

Schrödinger equation with constant mass is transformed to

$$\Psi''(x) + 2 \left( \frac{g'}{g} - \frac{f''}{2f'} \right) \Psi'(x) + \left\{ \left( \frac{g''}{g} - \frac{f''}{f'} \frac{g'}{g} \right) + 2(f')^2[\varepsilon - V(f(x))] \right\} \Psi(x) = 0. \tag{5}$$

By comparing each sides of the equations (2) and (5) term-by-term, we can identify the following conditions

$$g(x) = \left( \frac{f'(x)}{m(x)} \right)^{1/2} \quad (6)$$

and

$$E - V(x) = \frac{(f')^2}{m} [\varepsilon - V(f(x))] + \frac{1}{2m} F(f, g), \quad (7)$$

where  $F(f, g) = \left( \frac{g''}{g} - \frac{f''}{f'} \frac{g'}{g} \right)$ . If we consider the substitution map as  $(f')^2 = m$  in equation (7), our reference problem is transformed to the target problem including the energy spectra of the bound states, potential and wavefunction as

$$E_n = \varepsilon_n,$$

$$V(x) = V(f(x)) + \frac{1}{8m} \left[ \frac{m''}{m} - \frac{7}{4} \left( \frac{m'}{m} \right)^2 \right], \quad (8)$$

$$\Psi_n(x) = [m(x)]^{1/4} \Phi_n(f(x)).$$

Here, we point out that if we have a problem whose exact solution is well established, then we can apply the PCT method to this problem. That is, it preserves the structure of the wave equation of the target problem that has the same class as that of the reference problem.

### 3. Applications

In this section, we deal with three different position-dependent mass distributions. The reference potentials such as Morse, Pöschl-Teller, and Hulthén [35] are taken. Then, we are going to construct their target systems, respectively.

#### 3.1. Asymptotically vanishing mass distribution: $m(x) = \frac{\alpha^2}{x^2+q}$

The mapping function is

$$y = f(x) = \int^x \sqrt{m(x)} \, dx = \alpha \ell n \left( x + \sqrt{x^2 + q} \right) \quad (9)$$

with

$$x = \sinh_q \left( \frac{y}{\alpha} \right), \quad (10)$$

where  $\alpha \neq 0$ .

### 3.1.1. Morse case

Let us first consider the Morse potential as the reference problem [35, 36]

$$V(y) = D(e^{-2\alpha y} - 2e^{-\alpha y}) \equiv D \left[ (1 - e^{-\alpha y})^2 - 1 \right]. \quad (11)$$

This is the source potential with the energy eigenvalues and eigenfunctions as [35]

$$\varepsilon_n = -\frac{\alpha^2 \hbar^2}{2m} \left[ \bar{D} - \left( n + \frac{1}{2} \right) \right]^2, \quad (12)$$

$$\Phi_n(y) = C_n z^\beta e^{-\eta z} L_n^t(z)$$

with  $z = -e^{\alpha y}$ . By putting the mass function and equation (9) into the equation (8) and using (10), the target problem can be constructed as follows:

$$E_n = \varepsilon_n,$$

$$V(x) = D \left\{ \left[ 1 + \alpha^2 \left( x + \sqrt{x^2 + q} \right) \right]^2 - 1 \right\} + \frac{1}{8\alpha^2} \left[ 1 + \left( \frac{q}{x^2 + q} \right) \right], \quad (13)$$

$$\Psi_n(x) = C_{\bar{n}} (x^2 + q)^{-1/4} [f(x)]^\beta e^{-\eta f(x)} L_n^t(f(x)),$$

where  $C_{\bar{n}} = \sqrt{\alpha} C_n$ .

### 3.1.2. Pöschl-Teller case

Now, we will consider the potential as [35, 36]

$$V(y) = -\frac{U_0}{\cosh^2(\alpha y)} \equiv \frac{-4U_0}{(e^{\alpha y} + e^{-\alpha y})^2}, \quad U_0 = \lambda(\lambda - 1). \quad (14)$$

Our reference potential is now order that its energy spectra and wavefunction are given [35]

$$\bar{\varepsilon}_n = A^2 - \frac{\alpha^2 \hbar^2}{2m} \left[ -\left( n + \frac{1}{2} \right) + \frac{1}{2} \sqrt{1 + 4\gamma^2} \right], \quad (15)$$

$$\Phi_n(y) = C_n (1 - z^2)^{\beta/2} P_n^{(\beta, \beta)}(z)$$

with  $z = \tanh(\alpha y)$ . In this case, we want to construct the target problem for asymptotically vanishing mass distribution. Following the same procedure as in

above, we get the target system

$$\begin{aligned} E_n &= \bar{\varepsilon}_n, \\ V(x) &= -\frac{U_0}{x^2 + q} + \frac{1}{8\alpha^2} \left[ 1 + \left( \frac{q}{x^2 + q} \right) \right], \\ \Psi_n(x) &= C_{\bar{n}}(x^2 + q)^{-1/4} [1 - f^2(x)]^{\beta/2} P_n^{(\beta, \beta)}(f(x)), \end{aligned} \quad (16)$$

where  $C_{\bar{n}} = \sqrt{\alpha} C_n$ .

### 3.1.3. Hulthén case

Hulthén potential is defined [35, 36]

$$V(y) = -V_0 \frac{e^{-\alpha y}}{(1 - e^{-\alpha y})} \equiv -V_0 (e^{\alpha y} - 1)^{-1}. \quad (17)$$

Its energy spectrum and corresponding wavefunctions are

$$\begin{aligned} \tilde{\varepsilon}_n &= -V_0 \left[ \frac{\beta^2 - \bar{n}^2}{2\bar{n}\beta} \right]^2, \\ \Phi_n(y) &= C_n z^{\tilde{\varepsilon}} (1 - z)^{\mu/2} P_n^{(2\tilde{\varepsilon}, \mu-1)}(1 - 2z) \end{aligned} \quad (18)$$

with  $z = e^{-\alpha y}$  [35]. The target system is constructed with the mass function

$$\begin{aligned} E_n &= \tilde{\varepsilon}_n, \\ V(x) &= -V_0 \left[ -1 + \alpha^2 \left( x + \sqrt{x^2 + q} \right) \right]^{-1} + \frac{1}{8\alpha^2} \left[ 1 + \left( \frac{q}{x^2 + q} \right) \right], \\ \Psi_n(x) &= C_{\bar{n}}(x^2 + q)^{-1/4} [f(x)]^{\tilde{\varepsilon}} [1 - f(x)]^{\mu/2} P_n^{(2\tilde{\varepsilon}, \mu-1)}(1 - 2f(x)). \end{aligned} \quad (19)$$

### 3.2. Hyperbolic mass distribution I: $m(x) = \tanh_q^2(\alpha x)$

As a second example, we want to deal with the square hyperbolic mass functions. The resulting map for the mass type becomes

$$y = \bar{f}(x) = \frac{1}{\alpha} \ell n \cosh_q(\alpha x) \quad (20)$$

with  $x = \frac{1}{\alpha} \cosh_q^{-1}(e^{\alpha y})$ .

### 3.2.1. Morse case

For the case, the potential functions and the wavefunctions having the same spectrum are obtained as

$$\begin{aligned}
 V(x) &= D \left\{ [1 + \cosh_q(\alpha x)]^2 - 1 \right\} - \frac{\alpha^2/2}{\sinh_q^4(\alpha x)} \left[ \frac{5}{4} + \sinh_q^2(\alpha x) \right], \\
 \Psi_n(x) &= C_{\bar{n}} \sqrt{[\tanh_q(\alpha x)][\bar{f}(x)]^\beta} e^{-\eta \bar{f}(x)} L_n^t(\bar{f}(x)),
 \end{aligned}
 \tag{21}$$

where the hyperbolic functions are  $\cosh_q(x) = (e^x + qe^{-x})/2$ ,  $\sinh_q(x) = (e^x - qe^{-x})/2$  and  $\tanh_q(x) = \left( \frac{\sinh_q(x)}{\cosh_q(x)} \right)$  [37].

### 3.2.2. Pöschl-Teller case

By following the PCT procedure, the target system is constructed

$$\begin{aligned}
 V(x) &= -4U_0 [\cosh_q(\alpha x) + \operatorname{sech}_q(\alpha x)]^{-2} - \frac{\alpha^2/2}{\sinh_q^4(\alpha x)} \left[ \frac{5}{4} + \sinh_q^2(\alpha x) \right], \\
 \Psi_n(x) &= C_{\bar{n}} \sqrt{[\tanh_q(\alpha x)][1 - (\bar{f}(x))^2]^{\beta/2}} P_n^{(\beta, \beta)}(\bar{f}(x)),
 \end{aligned}
 \tag{22}$$

where  $\operatorname{sech}_q(x) = \frac{1}{\cosh_q(x)}$  [37]. Hence, the energy spectrum of the bound state is the same as that of the reference one.

### 3.2.3. Hulthén case

The mapping function leads to the following target system

$$\begin{aligned}
 V(x) &= -V_0 [-1 + \cosh_q(\alpha x)]^{-1} - \frac{\alpha^2/2}{\sinh_q^4(\alpha x)} \left[ \frac{5}{4} + \sinh_q^2(\alpha x) \right], \\
 \Psi_n(x) &= C_{\bar{n}} \sqrt{[\tanh_q(\alpha x)][\bar{f}(x)]^{\bar{\epsilon}}} [1 - \bar{f}(x)]^{\mu/2} P_n^{(2\bar{\epsilon}, \mu-1)}(1 - 2\bar{f}(x)).
 \end{aligned}
 \tag{23}$$

This system and its original one share the same spectrum.

## 3.3. Hyperbolic mass distribution II: $m(x) = \coth_q^2(\alpha x)$

If we consider the second type hyperbolic mass function, we also get

$$y = \tilde{f}(x) = \frac{1}{\alpha} \ell n \sinh_q(\alpha x),
 \tag{24}$$

with  $x = \frac{1}{\alpha} \sinh_q^{-1}(e^{\alpha y})$ .

### 3.3.1. Morse case

This mapping yields the potential function and the wavefunction with the same spectrum

$$V(x) = D \left\{ [1 + \sinh_q(\alpha x)]^2 - 1 \right\} - \frac{\alpha^2}{2} \left[ \frac{1}{\sinh_q^2(\alpha x)} + \frac{(9/4)}{\cosh_q^4(\alpha x)} \right], \quad (25)$$

$$\Psi_n(x) = C_{\bar{n}} \sqrt{[\coth_q(\alpha x)]} [\tilde{f}(x)]^\beta e^{-\eta \tilde{f}(x)} L_n^t(\tilde{f}(x)),$$

where  $\coth_q(x) = \left( \frac{\cosh_q(x)}{\sinh_q(x)} \right)$  [37].

### 3.3.2. Pöschl-Teller case

The target system with the same spectrum is then

$$V(x) = -4U_0 [\sinh_q(\alpha x) + \operatorname{cosech}_q(\alpha x)]^{-2} - \frac{\alpha^2}{2} \left[ \frac{1}{\sinh_q^2(\alpha x)} + \frac{(9/4)}{\cosh_q^4(\alpha x)} \right]$$

$$\Psi_n(x) = C_{\bar{n}} \sqrt{[\coth_q(\alpha x)]} [1 - (\tilde{f}(x))^2]^{\beta/2} P_n^{(\beta, \beta)}(\tilde{f}(x)), \quad (26)$$

where  $C_{\bar{n}} = \sqrt{\alpha} C_n$  and  $\operatorname{cosech}_q(x) = \frac{1}{\sinh_q(x)}$  [37].

### 3.3.3. Hulthén case

Finally, we construct the target system with the same energy spectrum of the bound state

$$V(x) = -V_0 [-1 + \sinh_q(\alpha x)]^{-1} - \frac{\alpha^2}{2} \left[ \frac{1}{\sinh_q^2(\alpha x)} + \frac{(9/4)}{\cosh_q^4(\alpha x)} \right],$$

$$\Psi_n(x) = C_{\bar{n}} \sqrt{[\coth_q(\alpha x)]} [\tilde{f}(x)]^{\tilde{\epsilon}} [1 - \tilde{f}(x)]^{\mu/2} P_n^{(2\tilde{\epsilon}, \mu-1)}(1 - 2\tilde{f}(x)) \quad (27)$$

with  $C_{\bar{n}} = \sqrt{\alpha} C_n$ .

## 4. Conclusions

In the present work we have applied the PCT approach for systems with some spatially dependent effective masses, exactly solvable potentials such as Morse, Pöschl-Teller, and Hulthén, whose bound-state energies and corresponding wavefunctions are determined algebraically. The determination of the mapping function plays an important role on the construction of the target system which has the exact closed forms of the energy spectrum and corresponding wavefunctions. In particular, the former system with the source potential and the latter one with the target potential will share the same bound-state spectra.

In this paper we use the deformation parameter  $q$  describing the mass distributions. For instance, when  $q = 1$ , the target potentials and wavefunctions of all potentials mentioned above are identical so that the energy spectra of the systems comply with that of the reference system [35]. We also point out as a final remark that both the form of the potential and spatial dependence of the mass  $m(x)$  cause a system to exactly solvability.

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